Axisymmetric, Self-Excited Oscillations in Parachutes

Bryan W. Roberts*
University of Sydney, Sydney, Australia

The paper derives the conditions under which self-excited, 'breathing' oscillations can exist in parachutes. The 'breathing' phenomenon is shown to be dependent on seven parameters. These parameters include the Froude number, the rigging line length to canopy length ratio, the steady drag coefficient, and the rigging line stiffness number. When a parachute is unstable in the 'breathing' mode it can be shown that the descent velocity fluctuations at the parachute store always lead the canopy fluctuations by exactly ($\pi/2$) rad. The paper concludes by stating the conditions that are necessary to ensure that a particular parachute will be stable in the 'breathing' mode. The system will always be stable if certain aerodynamic derivatives are related in the following manner. That is, for absolute stability

$$M_{10}M_3(N_{11} - N_4) < 0, \text{ when } M_4 = M_{11}$$

Nomenclature

C = unsteady drag coefficient
d_T = total damping inherent in the lines
g = acceleration due to gravity
h^2 = focal length of the approximating parabola
H = dimensionless form of h
K_T = total stiffness of the rigging lines
L = length of the canopy from apex to skirt
L' = rigging line length
M = mass of the store
p = pressure on the parabola's surface
V_o = steady decent velocity of the parachute
ϕ = phase of the drag perturbation
Φ = a velocity potential
μ = dimensionless descent velocity perturbation
η = dimensionless distance in the F-plane[2]
θ = phase of the velocity perturbation
Ω = dimensionless breathing frequency

I. Introduction

IT is often observed in certain parachute systems that the projected or inflated area of the parachute varies periodically in time. This phenomenon is often described as parachute "breathing" or "pumping" and is characterized by a periodic symmetric variation in the parachute's flying diameter.

Stevens and Hume[1] in a recent paper have made observations relating to conventional man-drop parachutes of the solid canopy type. They have observed, apart from the pendulum-type oscillations, that certain canopies appeared to "breathe" more violently than others. Furthermore, they found that the "breathing" frequency appeared to increase linearly with the mean descent speed of the system. (See, for instance, their Fig. 12 of Ref. 1).

In addition, they noted that "where the amplitude of the fluctuations in the descent velocity were high, the breathing and descent velocity fluctuations were strongly coupled. The coupling was not only in frequency but also in phase. The maximum velocity of descent appeared to occur about $\pi/2$ rad later than the minimum diameter of the canopy mount."

These authors, however, have made no examination of the mechanics relating to these self-excited, 'breathing' oscillations. The current paper is intended to predict from a theoretical basis that an instability can exist for a certain range of the appropriate parameters. Furthermore, it will be shown that when the system is unstable, the first harmonic velocity fluctuations at the parachute store do (as observed by Hume and Stevens) lead the first harmonic fluctuations in canopy diameter by exactly ($\pi/2$) radians.

Presented as Paper 73-452 at the AIAA 4th Aerodynamics Decleration Systems Conferences, Palm Springs, Calif., May 21–23, 1973; submitted July 20, 1973; revision received June 27, 1974.

Index categories: Nonsteady Aerodynamics; Entry Deceleration Systems and Flight Mechanics (e.g., Parachutes).

*Senior Lecturer, Department of Mechanical Engineering. Member AIAA.

II. Brief Dimensional Analysis of the "Breathing" Problem

In a dimensional study of the "breathing" phenomenon it will be assumed that the canopy has an average mass of ρ_c per unit of surface area and that the rigging lines below the skirt have a mass per unit of length of ρ_r.

Furthermore, it will be assumed that the canopy material is inelastic and nonporous, and that the canopy is attached to the store by a large number of rigging lines of length L', whose total stiffness is K_T. The effect of porosity can be included, as will be seen later, by suitably modifying the steady drag coefficient.

If the above system were to descend with complete stability at a velocity of V_o in an atmosphere of density ρ, under a gravitational field g, then, in the absence of compressibility effects and neglecting any change in the viscous effects (that is Reynolds number is invariant), any self-excited "breathing" oscillations will occur at a fundamental frequency of ω. This oscillation will also produce a fundamental velocity variation on the store of V_s.

It can be asserted that

$$\omega = f(V_o, L, \rho, g, K_T, d_T, M, L', \rho_c, \rho_r)$$

and

$$V_s = h(V_o, L, \rho, g, K_T, d_r, M, L', \rho_c, \rho_r)$$

If V_o, L, and M are chosen as the intrinsic variables then it follows that

$$\Omega = \left(\frac{\omega L}{V_o}\right) = f_1\left(\frac{L'}{L}, \frac{V_o^2}{gL}, \frac{\rho V_o^2 L^2}{Mg}, \frac{\rho V_o^2 L}{K_T}, \frac{d_T}{\rho V_o L^2}, \frac{\rho_c}{\rho L}, \frac{\rho_r}{\rho L^2}\right)$$

and

$$(V_s/V_o) = h_1\left(\frac{L'}{L}, \frac{V_o^2}{gL}, \frac{\rho V_o^2 L^2}{Mg}, \frac{\rho V_o^2 L}{K_T}, \frac{d_T}{\rho V_o L^2}, \frac{\rho_c}{\rho L}, \frac{\rho_r}{\rho L^2}\right)$$

Therefore, it is expected that any 'breathing' phenomenon in a particular type of canopy will be simply a function of seven parameters. These parameters are

a) rigging line length to canopy length ratio, L'/L
b) Froude number, V_o^2/gL
c) steady drag coefficient, $\rho V_o^2 L^2/Mg$
d) line stiffness number, $\rho V_o^2 L/K_T$
e) line damping ratio, $d_T/\rho V_o L^2$
f) canopy to atmosphere density ratio, $\rho_c/\rho L$
g) line to atmosphere density ratio, $\rho_r/\rho L^2$.

It now remains to postulate a theoretical model involving these seven parameters.

III. Unsteady Pressure Distribution on a "Breathing" Canopy

The unsteady pressure distribution on an impervious canopy can be derived using the method detailed in a previous paper on parachute inflation dynamics.[2] It is in fact useful to point out that the inflation-deflation of the canopy occurring during a "breathing" oscillation is a special, steady-state solution to the general problem of parachute inflation.

Proceed generally according to Ref. 2 by approximating the canopy to a parabolic shell of revolution having a focal length of h^2, where h is a function of time. It may be seen in Fig. 1 that this parabolic approximation to the canopy shape is satisfactory for parachutes in a symmetric "breathing" mode.

The pressure on the canopy surface may be calculated by deriving the velocity on the surface of the body in the F-plane.[2] This velocity in the F-plane, V_F, is comprised of two parts. The first part is due to the unsteady freestream flow, μV_o, approaching the canopy, where the dimensionless freestream velocity, μ, is a function of time only.

The second contribution to the flow in the F-plane is due to a vortex sheet buried in the inflating (or deflating) parabolic shell. Hence by addition, the total velocity, V_F, on the parabolic surface in the F-plane

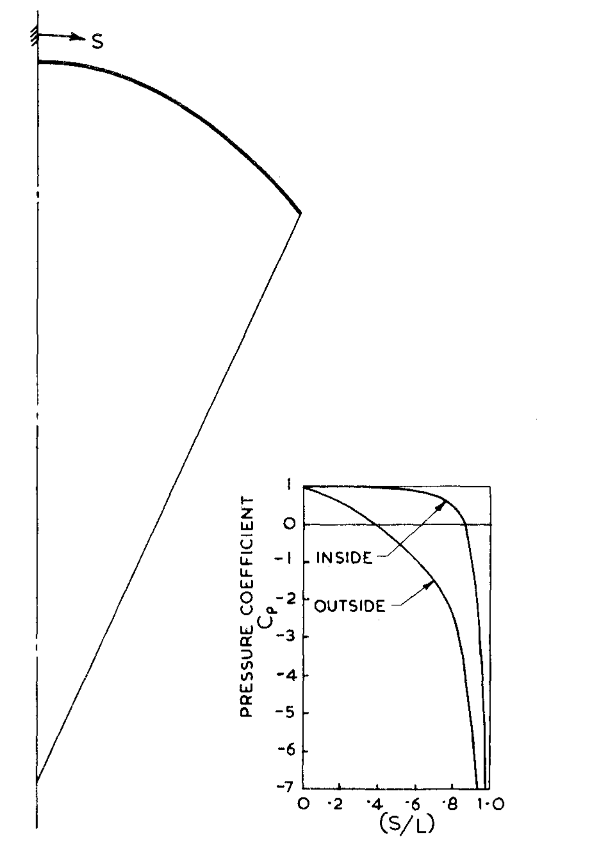

Fig. 1 Steady flying shape and steady pressure distribution $L'/L = 2, H = 1.79$.

$$\left(\frac{V_F}{V_o a}\right) = \mu\left(\frac{V_m}{V_o a}\right) + \frac{dH}{d\tau}\left(\frac{V_v}{V_o a}\right) \tag{1}$$

where the functions $(V_m/V_o a)$ and $(V_v/V_o a)$ are set out in Ref. 2. Therein, it can be seen that the terms contributing to V_F are functions of μ, η, H, and $dH/d\tau$.

The F-plane velocity may be used to calculate the velocity on the parabolic surface in the Z-plane by multiplying V_F by $(d\bar{F}/dZ \times dF/dZ)^{1/2}$. Hence the total velocity on the surface in the Z-plane is

$$\left(\frac{V_z}{V_o}\right) = \left(\frac{d\bar{F}}{dZ} \times \frac{dF}{dZ}\right)^{1/2}\left[\mu\left(\frac{V_m}{V_o a}\right) + \frac{dH}{d\tau}\left(\frac{V_v}{V_o a}\right)\right] \tag{2}$$

where $(d\bar{F}/dZ \times dF/dZ)^{1/2}$ is a function of η and H.

Before one can write the unsteady Bernoulli equation it is necessary to find expressions for the disturbance potential on the surface of the parabola. Again, this disturbance potential consists of two parts. The first part is due to the unsteady freestream flow, while the second part is due to the vortex sheet buried in the parabola. It has been shown[2] that

$$\frac{\Phi_F}{V_o a^2} = \mu\left(\frac{\Phi_m}{V_o a^2}\right) + \frac{dH}{d\tau}\left(\frac{\Phi_v}{V_o a^2}\right) \tag{3}$$

where $(\Phi_m/V_o a^2)$ and $(\Phi_v/V_o a^2)$ are as functions of η and H.

Equations (2) and (3) can be substituted into the steady Bernoulli equation whose general form in a moving frame of reference is

$$p = p_\infty - \rho\left[\frac{\partial \Phi_1}{\partial t} - V \times \nabla_1 \Phi_1 + \tfrac{1}{2}(\nabla_1 \Phi_1)^2\right] \tag{4}$$

The term p is the pressure at a point on the parabola's surface p_∞ is the static pressure at infinity (here taken as zero), and Φ_1 is the disturbance potential, and ∇_1 relates to spacial gradients in a moving frame of reference located in the focal point of the parabola.

With some algebraic work it can be shown that Eq. (4) gives pressures on the surface of the inflating/deflating, accelerating/decelerating parabola of the form

$$\begin{aligned}\frac{p}{\tfrac{1}{2}\rho V_o^2} = 2\Bigg[&-\left(\frac{a^2}{L}\right)\frac{d}{d\tau}\left\{\mu\left(\frac{\Phi_m}{V_o a^2}\right) + \frac{dH}{d\tau}\left(\frac{\Phi_v}{V_o a^2}\right)\right\} \\ &+ \mu^2/2 - \tfrac{1}{2}\left\{\frac{8(\eta-\eta^2)^{1/2}}{\cos\theta}\left(\frac{a^2}{L}\right)\frac{dH}{d\tau}\right\}^2 \\ &- \left(\frac{d\bar{F}}{dZ}\frac{dF}{dZ}\right)^{1/2}\Big/2\left\{\mu\left(\frac{V_m}{V_o a}\right) + \frac{dH}{d\tau}\left(\frac{V_v}{V_o a}\right)\right\}^2\Bigg]\end{aligned} \tag{5}$$

where

$$\cos\theta = 4[(\eta-\eta^2)/\{16(\eta-\eta^2) + H^2\}] \tag{6}$$

For convenience of subsequent work we will partition the right-hand side of Eq. (5) into four parts such that

$$T_1 = -\left(\frac{a^2}{L}\right)\frac{d}{d\tau}\left[\mu\left(\frac{\Phi_m}{V_o a^2}\right) + \frac{dH}{d\tau}\left(\frac{\Phi_v}{V_o a^2}\right)\right] \tag{7}$$

$$T_2 = \mu^2/2 \tag{8}$$

$$T_3 = -\tfrac{1}{2}\left[\frac{8(\eta-\eta^2)^{1/2}}{\cos\theta}\left(\frac{a^2}{L}\right)\frac{dH}{d\tau}\right]^2 \tag{9}$$

$$T_4 = \left(\frac{d\bar{F}}{dZ} \times \frac{dF}{dZ}\right)^{1/2}\Big/2\left[\mu\left(\frac{V_s}{V_o a}\right) + \frac{dH}{d\tau}\left(\frac{V_v}{V_o a}\right)\right]^2 \tag{10}$$

The form of the solution to Eq. (5) under steady flow conditions may be seen in Fig. 1. In the case shown $\mu = 1$ and all time derivatives are zero. Shown as an insert to Fig. 1 is the pressure coefficient distribution for the case $H = 1.79$. The pressure tends to $-\infty$ as S/L tends to unity, but it has been arbitrarily assumed that $C_p = 0$ at the tip, and that C_p decreases linearly to the potential solution in the range $0.94 < \eta < 1.0$.

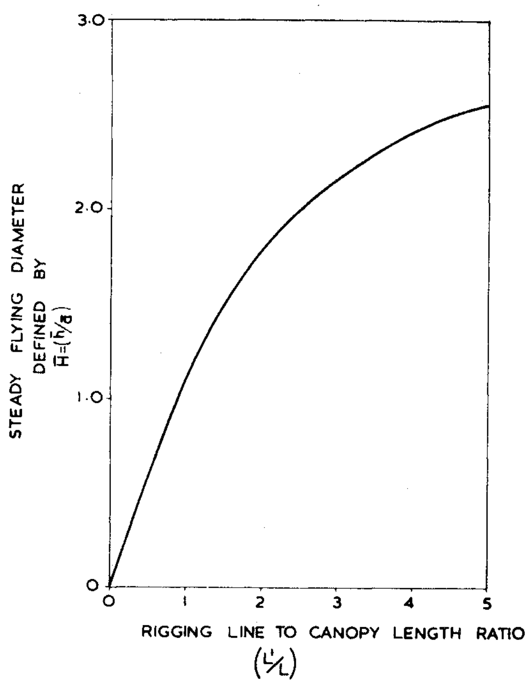

Fig. 2 Steady flying diameter.

The main section of Fig. (1) relates to a sectional elevation of the flying parabolic shell of revolution under the action of the pressure distribution shown in the insert. Under the conditions depicted, that is when $L'/L = 2$, the moments of all external forces on the parabola are zero. So that under steady flight conditions, when the moments about the apex are zero, H is uniquely determined as $\bar{H} = 1.79$ when $L'/L = 2.0$. Of course, as the lines are shortened the steady value of H is reduced and the pressure distribution is slightly modified. In fact, in Fig. 2 is shown the variation of the steady flying values of H, namely $\bar{H}$, for various values of L'/L. For all the cases in Fig. 2, the moment of all the external forces on the parabola are zero. Thus Fig. 2 effectively determines a series of unique values of $\bar{H}$ for a range of the variable (L'/L).

Returning to Eqs. (7)–(10) under time-varying conditions it may be seen that terms T_2 and T_3 have no contribution to the drag force or moment about the apex of the parabola. This is because both T_2 and T_3 are even functions of η about the $\eta = \frac{1}{2}$ position. Thus, by integration of the first and fourth terms (T_1 and T_4) of Eq. (5) one can find the unsteady drag coefficient and the unsteady moment coefficient about the apex.

Before completing these two integrations we will determine the various aerodynamic pressure derivatives associated with terms T_1 and T_4 in Eqs. (7) and (10).

IV. Unsteady Pressure Derivatives

The pressure derivatives associated with the terms T_1 and T_4 in the "breathing" mode may be found as follows.

Define a typical "breathing" oscillation of the form

$$H = \bar{H} + H_o + H_1 \cos \Omega\tau + \hat{H}_2 \cos(2\Omega + \alpha_2) + -- \tag{11}$$

where only the constant and fundamental frequency terms will be examined herein. Because of the "breathing," the freestream velocity variation will be of the form

$$\mu = \mu_o + \hat{\mu}_1 \cos(\Omega\tau + \theta_1) + \hat{\mu}_2 \cos(2\Omega\tau + \theta_2) + -- \tag{12}$$

where the constant term μ_o and the fundamental frequency terms $\hat{\mu}_1$ and θ_1 have yet to be determined.

Equations (11) and (12) may be substituted into Eq. (7) to give the first term T_1 as

$$T_1 = -\frac{a^2}{L}\left[\frac{d^2H}{d\tau^2}\left(\frac{\Phi_v}{V_o a^2}\right) + \left(\frac{dH}{d\tau}\right)^2 \times \frac{\partial}{\partial H}\left(\frac{\Phi_v}{V_o a^2}\right) + \mu\frac{dH}{d\tau}\times\frac{\partial}{\partial H}\left(\frac{\Phi_m}{V_o a^2}\right) + \frac{d\mu}{d\tau}\left(\frac{\Phi_m}{V_o a^2}\right)\right] \tag{13}$$

A series of A-coefficients may be defined corresponding to the various terms in (13) such that

$$A_1 = \frac{a^2}{L}\frac{\partial}{\partial H}\left(\frac{\Phi_v}{V_o a^2}\right)$$

$$A_2 = \frac{a^2}{L}\frac{\Phi_v}{V_o a^2}$$

$$A_3 = \frac{a^2}{L}\frac{\Phi_m}{V_o a^2} = A_5$$

$$A_4 = \frac{a^2}{L}\frac{\partial}{\partial H}\left(\frac{\Phi_m}{V_o a^2}\right) = 2A_6$$

The variation in the pressure coefficient, C_p, about its steady flow value can now be arranged in order of like time coefficients. Thus one obtains

Constant Variation

$$C_{p_o} = A_1\Omega^2\hat{H}_1{}^2 - A_6\mu_1\Omega\hat{H}_1 \sin\theta_1 \tag{14}$$

Cos$\Omega\tau$ Variation (In Phase)

$$C_{p_c} = A_2\Omega^2\hat{H}_1 + A_3\Omega\hat{\mu}_1 \sin\theta_1 \tag{15}$$

Sin$\Omega\tau$ Variation (Quadrature)

$$C_{p_s} = A_4\mu_o\Omega\hat{H}_1 + A_5\Omega\hat{\mu}_1 \cos\theta_1 \tag{16}$$

Next, similarly treat the fourth term, T_4, of Eq. (10) such that

$$T_4 = -\left(\frac{d\bar{F}}{dZ}\times\frac{dF}{dZ}\right)^{1/2}/2\left[\left(\mu\frac{V_s}{V_o a}\right)^2 + \left(\frac{dH}{d\tau}\right)^2\left(\frac{V_v}{V_o a}\right)^2 + 2\mu\frac{V_s}{V_o a}\times\frac{V_v}{V_o a}\right] \tag{17}$$

One can somewhat reduce the nonlinearity of Eq. (17) by making $(d\bar{F}/dZ \times dF/dZ)^{1/2}$ a function of η only.

$$\therefore \left(\frac{d\bar{F}}{dZ}\times\frac{dF}{dZ}\right)^{1/2} = f(\eta) \tag{18}$$

Continue by defining a series of D-coefficients such that

$$D_1 = \frac{f(\eta)}{2}\frac{\bar{V}_s{}^2}{V_o a} = D_6/2 = 2D_7$$

$$D_2 = \frac{f(\eta)}{2}\left\{\frac{\partial}{\partial H}\left(\frac{V_s}{V_o a}\right)\right\}^2$$

$$D_3 = \frac{f(\eta)}{4}\frac{\bar{V}_v{}^2}{V_o a}$$

$$D_4 = f(\eta)\left\{\frac{\partial}{\partial H}\left(\frac{V_s}{V_o a}\right)\times\frac{\bar{V}_s}{V_o a}\right\} = 2D_8$$

$$D_5 = f(\eta)\frac{\bar{V}_s}{V_o a}\times\frac{\bar{V}_v}{V_o a} = 2D_9$$

Similarly the D-coefficients produce a variation in the pressure coefficient as follows

Constant Variation

$$C_{p_o} = -D_1\mu_o^2 - D_7\hat{\mu}_1^2\cos^2\theta_1 - D_2\mu_o^2\hat{H}_1^2 - D_8\mu_o\hat{\mu}_1\cos\theta_1\hat{H}_1 - D_3\Omega^2\hat{H}_1^2 - D_7\hat{\mu}_1^2\sin^2\theta_1 + D_9\mu_1\sin\theta_1\Omega\hat{H}_1 \quad (19)$$

$\cos\Omega\tau$ Variation (In-Phase)

$$C_{p_c} = -D_6\mu_o\hat{\mu}_1\cos\theta_1 - D_4\mu_o^2\hat{H}_1 \quad (20)$$

$\sin\Omega\tau$ Variation (Quadrature)

$$C_{p_s} = -D_5\mu_o\Omega\hat{H}_1 + D_6\mu_o\hat{\mu}_1\sin\theta_1 \quad (21)$$

By addition of the like coefficients in Eqs. (14-16) and (19-21) one obtains the total variation in the pressure on the surface of the "breathing" parabola. It should be pointed out that as yet we have not calculated the form of the freestream velocity fluctuations as defined by Eq. (12). This calculation will now be performed in Sec. V.

V. Moment Criterion

In Sec. IV expressions have been found for the periodic variations in the pressure coefficient at the surface of the palpitating canopy. Implied in Eqs. (11) and (12) are a total of seven unknowns; namely $\bar{H}$, H_o, $\hat{H}_1$, Ω, μ_o, $\hat{\mu}_1$, θ_1. The first of these unknowns can be immediately determined from Fig. 2 once the value of L'/L is specified. In this section we will use three moment equations to solve for H_o, μ_1, and θ_1, while the remaining variables namely $\hat{H}_1$, Ω, and μ_o, will be found in a subsequent section.

To solve for H_o, $\hat{\mu}_1$, and θ_1 it will be asserted that the moments of all external forces on the canopy will be zero at all times. The apex will be chosen as a convenient origin for finding the external moments. In other words, the integration of the unsteady pressure distribution about the apex will be zero when due allowance is made for the mass of the canopy and the closing moment due to the rigging line tension.

In mathematical terms it follows that†

$$2048\int_0^1 C_p(1-2\eta)(\eta-\eta^2)^{3/2}d\eta + 256H^2\int_0^1 C_p(1-2\eta)(\eta-\eta^2)^{1/2}d\eta - \frac{128H^2\{(4a^2/L + L'^2/L^2 - 16a^4/L^2H^2)^{1/2}\}}{(L'^2/L^2 - 16a^4/L^2H^2)^{1/2}} \times \int_0^1 C_p(1-2\eta)\,d\eta + M_m = 0 \quad (22)$$

where the integrals of like coefficients throughout Eq. (22) will be all simultaneous zero. For example, if we define an integral operator I_1 such that

$$I_1(x) = 2048\int_0^1 x(1-2\eta)(\eta-\eta^2)^{3/2}d\eta + 256\,\bar{H}^2\int_0^1 x(1-2\eta)(\eta-\eta^2)^{1/2}d\eta \quad (23)$$
$$-\frac{128\bar{H}^2\{4a^2/L + (L'^2/L^2 - 16a^4/L^2\bar{H}^2)^{1/2}\}}{(L'^2/L^2 - 16a^4/L^2\bar{H}^2)^{1/2}}\int_0^1 x(1-2\eta)d\eta$$

then‡

†Mm is the inertia moment due to a finite mass of the canopy and lines.

‡M_{16} is the inertia moment coefficient.

Table 1 Numerical values of M- or N-coefficients

M or N Coefficient	Value $L'/L = 1$	$L'/L = 2$	$L'/L = 3$	$L'/L = 4$
M_1	−18.4	−23.05	−24.9	−26.7
M_2	−95.4	−87.1	−75.5	−68.1
M_3	41.4	74.4	93.4	109.2
M_4	26.0	32.5	35.1	37.7
M_5	41.4	74.4	93.4	109.2
M_6	13.0	16.3	17.6	18.8
M_7	0	0	0	0
M_8	0	0	0	0
M_9	−159.9	30.6	56.4	47.8
M_{10}	−120.0	−45.6	−34.7	−35.4
M_{11}	−123.1	−14.1	48.5	89.1
M_{12}	0	0	0	0
M_{13}	0	0	0	0
M_{14}	0	0	0	0
M_{15}	−61.5	−7.05	24.25	44.5
N_1	1.11	0.834	0.706	0.645
N_2	5.73	3.15	2.14	1.65
N_3	−2.49	−2.69	−2.65	−2.64
N_4	−1.56	−1.18	−0.997	−0.911
N_5	−2.49	−2.69	−2.65	−2.64
N_6	−0.782	−0.589	−0.498	−0.456
N_7	−3.21	−3.04	−2.77	−2.54
N_8	−0.133	−0.087	−0.067	−0.055
N_9	−14.5	2.22	3.44	2.66
N_{10}	0	0	0	0
N_{11}	−17.4	−3.46	2.14	4.91
N_{12}	−6.42	−6.08	−5.54	−5.08
N_{13}	−1.60	−1.52	−1.38	−1.27
N_{14}	−0.92	−0.73	−0.61	−0.53
N_{15}	−8.70	−1.73	1.07	2.45

$$I_1(C_{p_o}) = 0 \quad (24)$$

$$I_1(C_{p_c}) + M_{16}\Omega^2\hat{H}_1 = 0 \quad (25)$$

$$I_1(C_{p_s}) = 0 \quad (26)$$

Therefore, the three equations, Eqs. (24)–(26) inclusive, may be solved for the three unknowns H_o, $\hat{\mu}_1$, and θ_1 by substituting Eqs. (19)–(21), respectively. The appropriate coefficients, defined as M coefficients, appear in Eqs. (27)–(29), and their numerical values are shown in Table 1. After the appropriate substitutions described above one finds that

Constant Moment Variation

$$M_1\Omega^2\hat{H}_1^2 - M_6\hat{\mu}_1\Omega\hat{H}_1\sin\theta_1 - M_7\mu_o^2 - M_{13}\hat{\mu}_1^2\cos\theta_1 - M_8\mu_o^2\hat{H}_1^2 - M_{14}\mu_o\hat{\mu}_1\hat{H}_1\cos\theta_1 - M_9\Omega^2\hat{H}_1^2 - M_{13}\hat{\mu}_1\sin^2\theta_1 + M_{15}\hat{\mu}_1\Omega\hat{H}_1\sin\theta_1 - M_{10}\mu_o^2H_o = 0 \quad (27)$$

$\cos\Omega\tau$ Moment Variation

$$M_2\Omega^2\hat{H}_1 + M_3\Omega\hat{\mu}_1\sin\theta_1 - M_{12}\mu_o\mu_1\cos\theta_1 - M_{10}\mu_o^2\hat{H}_1 + M_{16}\Omega^2\hat{H}_1 = 0 \quad (28)$$

$\sin\Omega\tau$ Moment Variation

$$M_4\mu_o\Omega\hat{H}_1 + M_5\Omega\hat{\mu}_1\cos\theta_1 - M_{11}\mu_o\Omega\hat{H}_1 + M_{12}\mu_o\hat{\mu}_1\sin\theta_1 = 0 \quad (29)$$

The values of the M-coefficients for four values of L'/L are shown below in Table 1, and the value of M_{16} will be taken as zero.

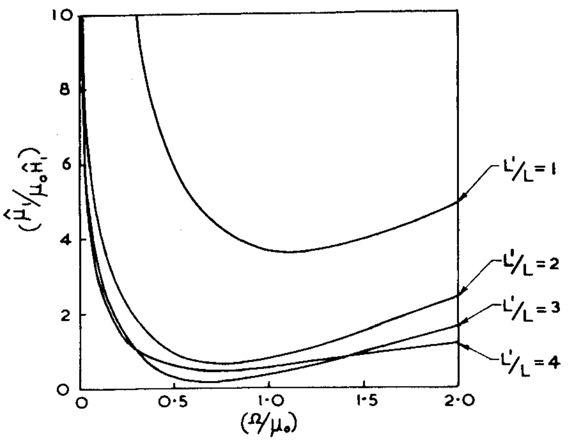

Fig. 3 Velocity fluctuation as a function of frequency.

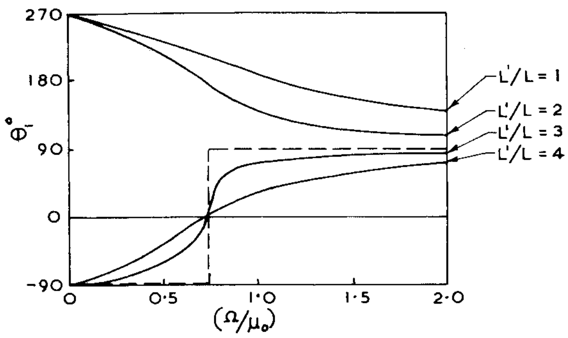

Fig. 4 Phase of the velocity fluctuations.

We will not bother with the formal solution of Eq. (27) because the variable H_o follows immediately once values for μ_0, $\hat{\mu}_1$ etc. are known. In any case H_o will be small as Eq. (27) contains products of small quantities throughout.

Now Eqs. (28) and (29) can be divided by $\mu_0\hat{H}_1$ and the zero M-coefficients substituted to give

$$\frac{\hat{\mu}_1}{\mu_o\hat{H}_1} \times \sin\theta_1 = \frac{M_{10} - M_2\left(\dfrac{\Omega}{\mu_o}\right)^2}{M_3\dfrac{\Omega}{\mu_o}} \tag{30}$$

and

$$\frac{\hat{\mu}_1}{\mu_o\hat{H}_1} \times \cos\theta_1 = \frac{M_{11} - M_4}{M_5} \tag{31}$$

From Eq. (30) and (31) the values of $(\hat{\mu}_1/\mu_0\hat{H}_1)$ and θ_1 can be found as functions of Ω/μ_0 and L'/L. The values of these dependant variables are shown in Figs. 3 and 4 and it is only this form of $(\hat{\mu}_1/\mu_0 H_1)$ and θ_1 which will ensure that the external moments about the apex are zero at all times. In Fig. 3 it may be seen that $(\hat{\mu}_1/\mu_o\hat{H}_1)$ has minimum values in the vicinity of (Ω/μ_0) equals 0.7, and that here is one value of (L'/L) which gives $\hat{\mu}_1/\mu_0 H_1$ equal to zero. The next step in the analysis is to examine the drag integrals, and thereby determine unique values of (Ω/μ_o) for which steady-state 'breathing' oscillations are possible.

VI. Unsteady Drag Force

Equations (14)–(16) and Eqs. (19)–(21) may be added and the result integrated to determine the unsteady drag coefficient of the 'breathing' parachute.

Define a drag coefficient, C, such that

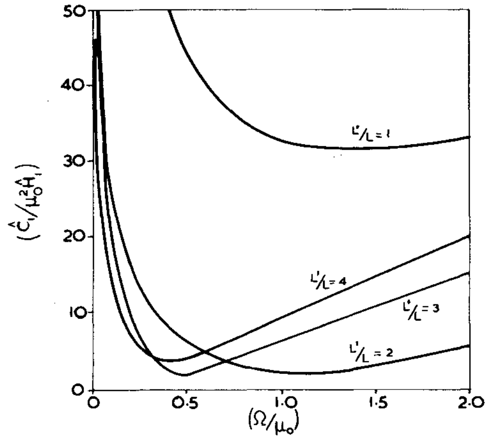

Fig. 5 Unsteady drag coefficient.

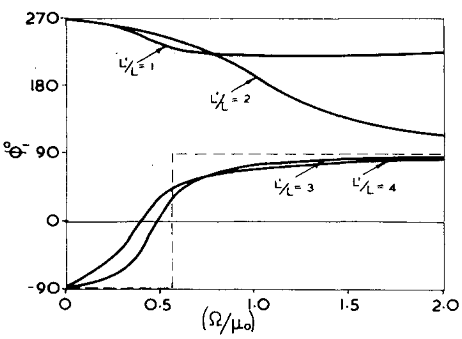

Fig. 6 Phase of the drag force variation.

$$C = \bar{C} + C_o + \hat{C}_1\cos(\Omega + \phi_1) + \hat{C}_2\cos(2\Omega + \phi_2) + \ldots \tag{32}$$

where only the constant and fundamental frequency terms will be considered hereafter.

Next it is possible to define in drag coefficient integral $I_2(x)$ such that

$$I_2(x) = \int_0^1 x(1 - 2\eta)d\eta, \tag{33}$$

and then the like pressure coefficients when integrated according to Eq. (33) will equal the like coefficients in Eq. (32). Thus one obtains

$$\bar{C} + C_o = I_2(C_{p_0}) \tag{34}$$

$$\hat{C}_1\cos\phi_1 = I_2(C_{p_c}) + N_{16}\Omega\hat{\mu}_1\sin\theta_1 \tag{35}$$

$$-\hat{C}_1\sin\phi_1 = I_2(C_{p_s}) + N_{16}\Omega\hat{\mu}_1\cos\theta_1 \tag{36}$$

Equations (34)–(36) may now be expanded to give Eqs. (37)–(39), where the N-coefficients are also shown in Table 1.

Constant Drag Variation

$$\bar{C} + C_o = N_1\Omega^2\hat{H}_1^2 - N_6\hat{\mu}_1\Omega\hat{H}_1\sin\theta_1 - N_7\mu_o^2 - N_{13}\hat{\mu}_1^2\cos^2\theta_1 - N_8\mu_o^2\hat{H}_1^2 - N_{14}\mu_o\hat{\mu}_1\hat{H}_1\cos\theta_1 - N_9\Omega^2\hat{H}_1^2 - N_{13}\hat{\mu}_1^2\sin^2\theta_1 + N_{15}\hat{\mu}_1\Omega\hat{H}_1\sin\theta_1 - N_{10}\mu_o^2 H_o \tag{37}$$

§N_{16} is the inertia drag coefficient due to a finite mass of the canopy and lines.

$\cos\Omega\tau$ Drag Variation

$$\hat{C}_1 \cos\phi_1 = N_2\Omega^2\hat{H}_1 + N_3\Omega\hat{\mu}_1 \sin\theta_1 - N_{12}\mu_o\hat{\mu}_1 \cos\theta_1 + N_{10}\mu_o{}^2\hat{H}_1 + N_{16}\Omega\hat{\mu}_1 \sin\theta_1 \qquad (38)$$

$\sin\Omega\tau$ Drag Variation

$$-\hat{C}_1 \sin\phi_1 = N_4\mu_o\Omega\hat{H}_1 + N_5\Omega\hat{\mu}_1 \cos\theta_1 - N_{11}\mu_o\Omega\hat{H}_1 + N_{12}\mu_o\hat{\mu}_1 \sin\theta_1 + N_{16}\Omega\hat{\mu}_1 \cos\theta_1 \qquad (39)$$

Now Eqs. (38) and (39) can be solved for $(\hat{C}_1/\mu_0{}^2\hat{H}_1)$ and ϕ_1 using the values of $(\hat{\mu}_1/\mu_0{}^2\hat{H}_1)$ and θ_1 from Figs. 3 and 4 where required. The values of $(\hat{C}_1/\mu_0{}^2\hat{H}_1)$ and ϕ_1 are shown in Figs. 5 and 6.

In the case of Eq. (37) it may be seen that $\bar{C}$ and C_0 are included as a summation. However, it is known from the form of Eq. (37) that $\bar{C} = -N_7$ and that $C_o = N_7(\mu_0{}^2 - 1)$ plus the remaining terms in Eq. (37). We will not bother at this stage to calculate C_o, but an interested reader could verify that C_o will be small.

VII. Kinematic Description of the Motion of the Store

It is at this point that the velocity of the store will be calculated using Fig. 7. It will be noted that we have expressions for $\hat{\mu}_1$ and θ_1, but the velocity of the store will be somewhat different to $\hat{\mu}_1$, the velocity of the canopy, due to rigging line stretch in combination with a change in the canopy diameter via the parameter $\hat{H}_1$.

A. Velocity of the Store Due to the dH/dt Term

In Fig. 7 a point C at the canopy skirt is moving downwards at a velocity μV_o and outwards at a velocity of V_c, where V_c is given below.

$$V_c + \frac{d(4a^2H)}{dt} = \frac{d(4a^2H)}{dH} \times \frac{dH}{dt} \qquad (40)$$

Therefore, by addition, the velocity of the store, V_s is

$$V_s = \mu V_o - V_{s_2} \qquad (41)$$

where V_{s2} is calculated by taking I_{cs} as an instantaneous center. Thus after some algebraic work it can be shown that

$$V_s = -x\left(\frac{V_o}{L}\right)\frac{dH}{dt} + \mu V_o \qquad (42)$$

where x is shown in Table 2 and is defined by Eq. (43)

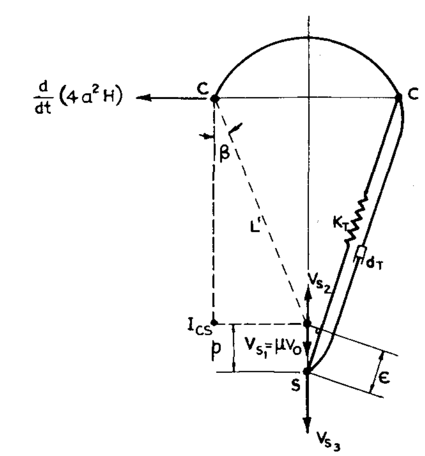

Fig. 7 Kinematic model of the parachute.

Table 2 Values of x

Function	$L'/L = 1$	$L'/L = 2$	$L'/L = 3$	$L'/L = 4$
L	0.317	0.064	0.023	0.016
T	2.00	1.23	1.10	1.06

$$x = \frac{4d(a^2H)}{dH} \times \frac{4a^2\bar{H}}{(L'^2 - 16a^4\bar{H}^2)^{1/2}} \qquad (43)$$

B. Velocity of the Store Due to Line Flexibility and Internal Line Damping

Referring to Fig. 7 it may be seen that another velocity perturbation at the store is due to line flexibility and line damping, K_T and d_T, respectively. Under the action of a drag force variation at the fundamental frequency, Ω, the lines will deflect by an amount ϵ. This line extension ϵ will cause the store to move downwards by an amount p. One can write the equation governing the line deflection as

$$K_T\epsilon + d_T\frac{d\epsilon}{dt} = QV_o{}^2\hat{C}_1 \cos(\Omega\tau + \phi_1)\sec\beta, \qquad (44)$$

where $Q = 8\pi\rho\tilde{a}^4\bar{H}^2$.

From geometric consideration it is known that

$$p \simeq \epsilon \sec\beta \qquad (45)$$

and substitution of Eq. (45) into (44) gives

$$K_Tp + d_Tdp/dt = QV_o{}^2\hat{C}_1 \cos(\Omega\tau + \phi_1)\sec^2\beta \qquad (46)$$

To solve for p, let $p = p_1 \cos(\Omega\tau + \psi_1)$ and substitute into Eq. (46) to give

$$p = \frac{QTV_o\,\hat{C}_1}{\left\{K_T{}^2 + d_T{}^2\Omega^2\left(\frac{V_o}{L}\right)^2\right\}^{1/2}} \cos(\Omega\tau + \psi_1) \qquad (47)$$

where

$$\psi_1 = \phi_1 - \tan^{-1}\left[\frac{d_TV_o\Omega}{K_TL}\right] \qquad (48)$$

and

$$T = \sec^2\beta = \frac{L'^2}{(L'^2 - 16a^4\bar{H}^2)} \qquad (49)$$

Finally, Eq. (47) can be differentiated with respect to time to give

$$V_{s_3} = dp/dt = -\frac{QTV_o{}^2\hat{C}_1\Omega}{\left\{K_T{}^2 + d_T{}^2\Omega^2\left(\frac{V_o}{L}\right)^2\right\}^{1/2}} \sin(\Omega\tau + \psi) \qquad (50)$$

where the values for T are given in Table 2.

C. Absolute Acceleration of the Store

First, the absolute velocity of the store is simply the sum of V_{s1}, V_{s2}, V_{s3}; therefore

$$V_s = \mu V_o + x\left(\frac{V_o}{L}\right)\Omega\hat{H}_1 \sin\Omega\tau - \frac{TQV_o{}^2\Omega C_1}{\left\{K_T{}^2 + d_T{}^2\Omega^2\left(\frac{V_o}{L}\right)^2\right\}^{1/2}} \sin(\Omega + \psi) \qquad (51)$$

Second, the absolute acceleration of the store can be found by differentiating Eq. (51) to give

$$\frac{1}{g}\left(\frac{dV_s}{dt}\right) = -\left(\frac{V_o^2}{gL}\right)\Omega\hat{\mu}_1 \sin(\Omega\tau + \theta_1) + \left(\frac{V_o^2}{gL}\right)\frac{x}{L}\Omega^2\hat{H}_1 \cos\Omega\tau - T\left(\frac{V_o^2}{gL}\right) \times \frac{QV_o^2}{K_TL}\frac{\dfrac{\Omega^2\hat{C}_1}{(1 + d_T^2\Omega^2V_o^2)^{1/2}}}{K_T^2L^2}\cos(\Omega\tau + \phi_1) \quad (52)$$

VIII. Store Kinetics

The final step in this treatment of the 'breathing' problem is to satisfy the equation of motion of the store. Newton's second law may be applied to the store to give

$$M\frac{dV_s}{dt} = Mg - 8\pi V_o^2\rho\bar{a}^4\bar{H}^2\{\mu_o^2(\bar{C} + C_o) + \hat{C}_1 \cos(\Omega\tau + \phi_1)\} \quad (53)$$

Equation (53) can be nomialised by dividing through by Mg, and by defining a variable Q as before. It then follows from Eq. (53) that

$$\frac{1}{g}\left(\frac{dV_s}{dt}\right) = 1 - \frac{QV_o^2}{Mg}\{\mu_o^2(\bar{C} + C_o) + \hat{C}_1 \cos(\Omega\tau + \phi_1)\} \quad (54)$$

Equation (52) can now be substituted into Eq. (54), and like coefficients equated to give

Constant Force Variation

$$\left(\frac{QV_o^2}{Mg}\right)\mu_o^2(\bar{C} + C_o) - 1 = 0 \quad (55)$$

$\cos\Omega\tau$ Force Variation

$$\frac{V_o^2}{gL}\left[-\Omega\hat{\mu}_1 \sin\theta_1 - T\frac{QV_o^2}{K_TL}\frac{\Omega^2}{\eta}\hat{C}_1 \cos(\phi_1 - \beta) + (x/L)\Omega^2\hat{H}_1\right] + \left(\frac{QV_o^2}{Mg}\right)\hat{C}_1 \cos\phi_1 = 0 \quad (56)$$

$\sin\Omega\tau$ Force Variation

$$\frac{V_o^2}{gL}\left[-\Omega\hat{\mu}_1 \cos\theta_1 + T\frac{QV_o^2}{K_TL}\frac{\Omega^2}{\eta}\hat{C}_1 \sin(\phi_1 - \beta)\right] - \left(\frac{QV_o^2}{Mg}\right)\hat{C}_1 \sin\phi_1 = 0 \quad (57)$$

where

$$\eta = \left\{1 + \frac{d_T^2\Omega^2V_o^2}{K_T^2L^2}\right\}^{1/2}$$

and

$$\beta = \tan^{-1}[d_TV_o\Omega/K_TL]$$

It should be noted that $\bar{C} = Mg/QV_o^2$, and that $\bar{C} = -N_7$ as noted earlier. Therefore, from Eq. (55) μ_o can be calculated. However, μ_o will be of the order of unity, and we will not bother further with solving explicitly for μ_o.

From Eq. (56) and (57) it is possible to solve for (V_o^2/gL) and Ω once the line stiffness parameter (QV_o^2/K_TL) and the line damping parameter $(d_T/\rho V_oL^2)$ have been specified. These solutions (as functions of the Froude number, and the reduced frequency parameter) will be the conditions for which steady-state breathing oscillations may exist.

IX. Solutions to Eqs. (56) and (57) for Zero Line Damping

The simplest solution to these equations is when the damping parameter is absent. Thus, for the moment assume that $\eta = 1, \beta = 0$.

A. Particular Solution with Zero Line Damping

A particular solution to Eqs. (56) and (57) occurs when

$$\sin\phi_1 = \cos\theta_1 = 0$$

so that Eq. (57) is automatically satisfied. From Fig. 4 it can be seen that $\theta_1 = \pm\pi/2$ when $(L'/L) = (L'/L)^* = 2.70$. The asterisked values refer to the steady-state, "breathing" conditions. It should be noted that this condition for θ_1 is automatically satisfied for all values of (Ω/μ_o). Thus (Ω/μ_o) is at the moment left open.

From Fig. (6) it may be seen that $\phi_1 = 0$ when $(\Omega/\mu_o)^* = 0.640$ and $(L'/L) = 2.70$. Therefore, Eq. (57) is satisfied when

$$(L'/L)^* = 2.70$$
$$(\Omega/\mu_o)^* = 0.64$$

Next, turn to Eq. (56). For the asterisked conditions it can be found from the previous work that

$$\{(\hat{\mu}_1/\mu_o\hat{H}_1)\sin\theta_1\}^* = -0.12$$

and

$$\{(\hat{C}_1/\mu_o^2\hat{H}_1)\cos\phi_1\}^* = 0.65$$

These latter values can now be substituted into Eq. (56) along with the appropriate values for $(x/L)^*$ and T^* to give

$$N_F = 1.76/\{2.35(\mu_o^2N_s) - 1\} \quad (58)$$

where

N_F, Froude Number $= (V_o^2/gL)^*$

and

N_s, Line Stiffness Number $= (QV_o^2/K_TL)$

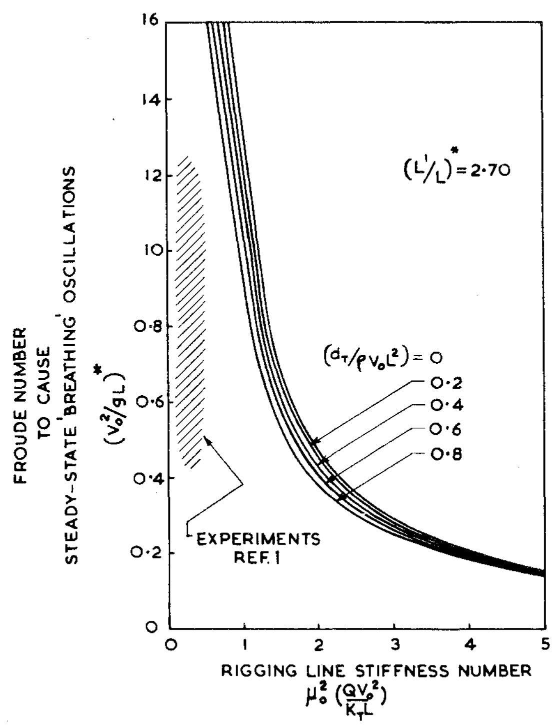

Fig. 8 Stable phase relationships.

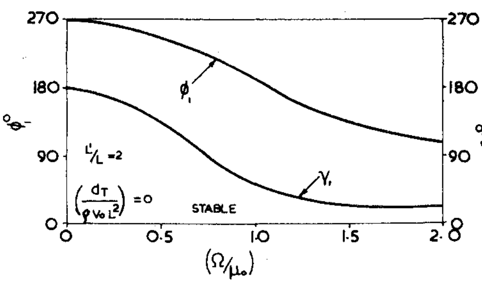

Fig. 9 Unstable phase relationships.

From Eq. (58) it is possible to construct Fig. 8 where it may be seen that *steady-state* "breathing" oscillation can only exist when the conditions along the upper curve of Fig. 8 are satisfied. Conditions differing from the curve will be damped due to aerodynamic terms inherent in Eq. (57).

B. General Solution for Zero Line Damping

The general solution to Eqs. (56) and (57) can be best determined by taking all the ϕ_1 terms to the right-hand side of their respective equations, and then dividing Eq. (57) by Eq. (56). In this manner it can be shown that steady "breathing" can only exist when

$$\phi_1 = \gamma_1 \tag{59}$$

where

$$\gamma_1 = \tan^{-1}\left[\frac{-(\hat{\mu}_1/\mu_o\hat{H}_1)\cos\theta_1}{(\hat{\mu}_1/\mu_o\hat{H}_1)\sin\theta_1 - (x/L)(\Omega/\mu_o)}\right] \tag{60}$$

The ϕ_1 and γ_1 functions are shown in Figs. 9 to 10 for the cases $L'/L = 2$, 2.7, respectively. In all cases, except the case $L'/L = 2.70$, there is no steady solution. Yet when $L'/L = 2.70$, the only solution is the particular solution already discussed in Sec. VIII. One can therefore conclude that in the zero line damping case there is one unique solution corresponding to $\gamma_1 = 0$, which is equivalent to the case $\theta_1 = \pi/2$.

X. Broadening Effect of Line Damping

A. Particular Solution for Finite Damping

The conclusions reached in Sec. IX-A above will now be modified to include the case of finite rigging line damping. One can proceed by seeking a solution to Eqs. (56) and (57) when η and β are both nonzero and positive. β will be taken as small.

In this case it will be realized that θ_1 and $\hat{\mu}_1/\mu_o\hat{H}_1$ will be unchanged, while ϕ_1 and $\hat{C}_1/\mu_0{}^2\hat{H}_1$ will adopt new values. With these notions one can write that

$$\cos\theta_1 = 0, \text{ as before,}$$
$$\eta \simeq 1$$
$$\sin\phi_1 \simeq \phi_1$$
$$\sin\beta \simeq \beta$$

and

$$(\Omega/\mu_o) \simeq 0.64$$

provided β is small. Therefore, Eq. (57) may be written as

$$-\frac{1}{\bar{C}}\phi_1 + TN_F(N_s\mu_o{}^2)(0.64)^2\{\phi_1 - \beta\} = 0 \tag{61}$$

If we now consider a small perturbation $\Delta(\Omega/\mu_o)$ in (Ω/μ_o) due the inclusion of the β term then

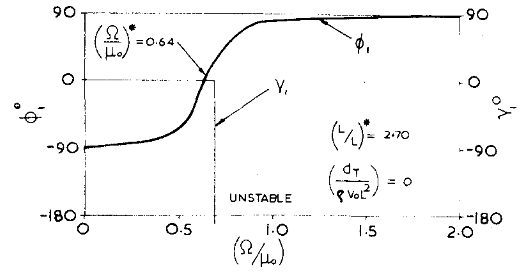

Fig. 10 Froude number to cause "breathing."

$$\frac{1}{\bar{C}}\frac{\partial\phi_1}{\partial(\Omega/\mu_o)}\Delta(\Omega/\mu_o) + 0.41\,TN_F(N_s\mu_o{}^2)\left\{\frac{\partial\phi_1}{\partial(\Omega/\mu_o)}\Delta(\Omega/\mu_o) - \beta\right\} = 0 \tag{62}$$

Hence it follows that

$$\Delta(\Omega/\mu_o) = \frac{0.41\,TN_F(N_s\mu_o{}^2)\beta}{\frac{\partial\phi_1}{\partial(\Omega/\mu_o)}\left\{0.41\,TN_F(N_s\mu_o{}^2) - \frac{1}{\bar{C}}\right\}} \simeq \frac{\beta}{\left\{\frac{\partial\phi_1}{\partial(\Omega/\mu_o)}\right\}} \tag{63}$$

Turning now to Eq. (56) with the appropriate approximations and substitutions it can be shown that

$$N_F\{0.64 \times 0.12 + (x/L)(0.64)^2\} - 0.41\,TN_F(N_s/\mu_o{}^2)0.65\{1 + \tan\phi_1\sin\beta\} + (0.65/\bar{C}) = 0 \tag{64}$$

and hence

$$N_F{}^* = \frac{1.76}{[2.35(1 + \phi_1\beta)\mu_o{}^2N_s - 1]} \tag{65}$$

From Eq. (61) is follows that

$$\phi_1\beta = -\frac{0.41\,TN_F(N_s\mu_o{}^2)\beta^2}{\left\{0.41\,TN_F(N_s\mu_o{}^2) - \frac{1}{\bar{C}}\right\}} \simeq \beta^2 \tag{66}$$

One can now substitute Eq. (66) into (65) to give

$$N_F{}^* = \frac{1.76}{[2.35(1 + \beta^2)\mu_o{}^2N_s - 1]} \tag{67}$$

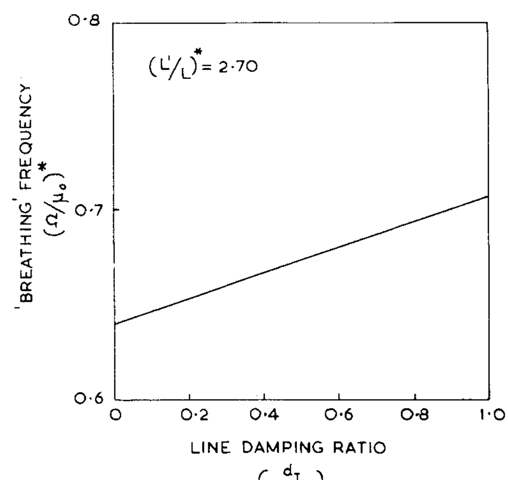

Fig. 11 "Breathing" frequency.

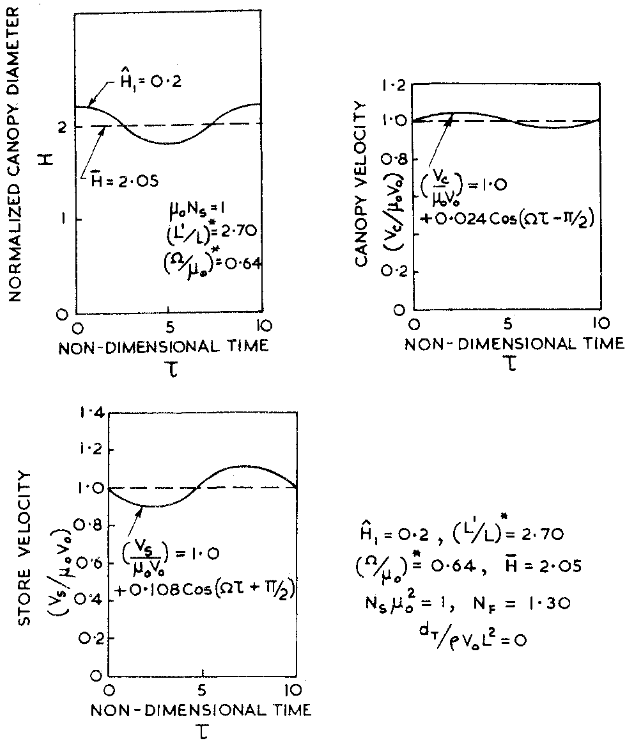

Fig. 12 A typical breathing oscillation.

Equations (67) and (63) are shown graphically in Figs. 8 and 11, respectively, as functions of the line damping parameter $(d_T/\rho V_o L^2)$. In Fig. 8 it can be seen that the effect of damping is to broaden the conditions for steady-state "breathing" oscillations. Also included in Fig. 8 are the experimental results from Ref. 1. In Fig. 11 the effect of damping is to produce a slight upward shift in the "breathing" frequency of about 10%.

B. General Solution for Finite Damping

When preparing this paper, it was felt that no positive statement could be made as to whether the above particular solution, with damping included, was in fact a unique solution to Eqs. (56) and (57). It is preferred at the moment to leave this question of uniqueness open.

XI. Conclusions

The conclusions to be drawn from the above work are as follows.

a) A 'breathing' parachute is essentially a two-degree-of-freedom system involving the vertical motion of two masses. One mass is the virtual mass of the airflow about the canopy, plus the mass of the canopy itself if it is significant. The second mass is the store. Both masses are connected by a series of slightly flexible rigging lines.

b) In this two-degree-of-freedom system, the zero frequency or first mode oscillation, corresponds to the steady descent of the parachute.

c) The second mode of oscillation is in fact the 'breathing' mode. This mode can only exist *in the steady-state* when Eq. (31) is identically zero, namely when $M_4 = M_{11}$.

d) An additional constraint on the existence of a "breathing" mode is that it can only occur at a particular Froude number. This Froude number is determined uniquely by two parameters, the line stiffness number and the line damping ratio (see Fig. 8).

e) In the case of zero line damping the 'breathing' frequency is given by

$$\left(\frac{\Omega}{\mu_o}\right)^* = \left(\frac{N_{12}M_{10}}{M_2N_{12} + M_3(N_{11} - N_4)}\right)^{1/2}$$

f) Line damping tends to increase the 'breathing' frequency slightly (see Fig. 11).

g) Typical time variations of the canopy diameter, store velocity and canopy velocity are shown in Fig. 12. In all cases the canopy diameter phasor will lead the canopy velocity phasor by exactly $\pi/2$ rad.

In addition, the store velocity phasor will lead the canopy diameter phasor by exactly $\pi/2$ rad. This latter phenomenon has been observed in practice and is reported in Ref. 1.

h) In no case was it possible to calculate the absolute amplitude of the oscillations, namely $\hat{H}_1$. The absolute amplitudes are limited by the nonlinearities in the system, which are not included in the current theory.

i) All aerodynamic derivatives that have been used in the calculation have been derived from a theoretical basis. This has led to minor unrealities. For instance, the canopy virtual mass has been overestimated and the 'breathing' reduced frequency is somewhat lower than that observed in practice.

j) There is a need for an experimental determination of the aerodynamic derivatives using realistic wake flows. In this way more realistic values of $(L'/L)^*$ and $(\Omega/\mu_o)^*$ can be evaluated. It is essential that all the N and M coefficients be found experimentally.

k) The most effective means of ensuring stability is to suitably modify the canopy geometry and/or airflow so that $M_{10}M_3(N_{11} - N_4) < 0$ when $M_4 = M_{11}$.

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